Continuation in Experiments

Lecture given during
Advanced Summer School on
Continuation Methods for Nonlinear Problems

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CRITICS
Critical Transitions in Complex Systems

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Outline

- basic idea
- experiments (mechanical)
- simulations (random dynamic network problem)

Experimental results

- (forced mechanical oscillators) DAW Barton (Univ. Bristol)
- (forced mechanical oscillators) DTU group (J Starke, F Schilder, E Bureau, JJ Thomsen, I Santos)
- (static bucking) Neville et al. (Bristol)

General terms and conditions

To do (in experiments)

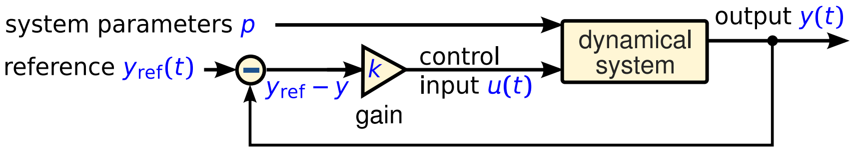
- Continue equilibria & periodic orbits that are either
  - dynamically unstable or
  - depend sensitively on system parameters

Constraints

- no setting of internal state possible
- accuracy independent of model
- good estimate for error
- avoid system identification
- no real-time computations

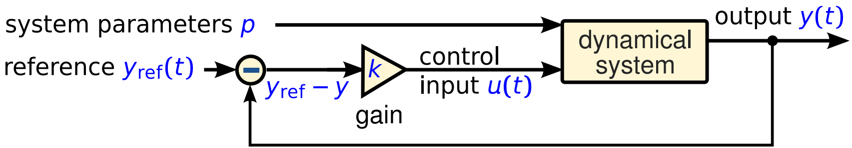
Basic idea

Feedback control



Basic idea

Feedback control



Classical control engineering

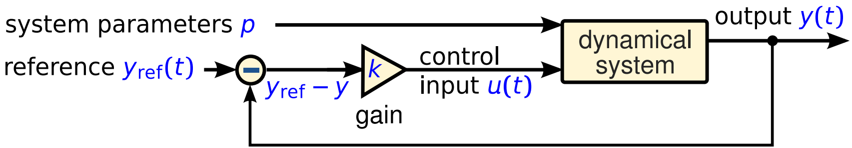
- reference $y_{ref}(t)$ given
- make output y track $y_{ref}(t)$

⇒ (e.g.) integral components in "k"

- Questions: stability, optimality, robustness,. . .

Basic idea

Feedback control



Find equilibria

$$\frac{d}{dt} y_{ref} = k_w [y - y_{ref}]$$

- washout filter (Abed *et al* 2004)

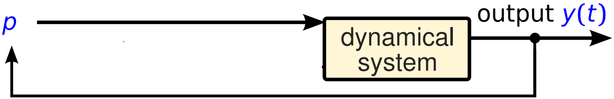
⇒ equilibria with control = equilibria without control

- suppresses Hopf, period doubling

Basic idea

Feedback control

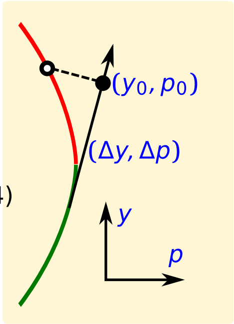
system parameters p



Find equilibria

$$\frac{\mathrm{d}}{\mathrm{d}t}q = \Delta y^T(y - y_o) + \Delta p(p - p_0)$$

$$p = p_{\mathrm{ref}} + [k_y, k_p]\begin{bmatrix} y - y_0 \\ q \end{bmatrix}$$



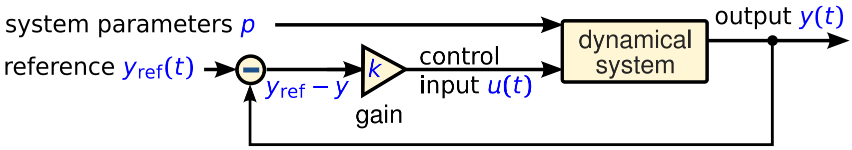
- control through parameter p (Siettos *et al* 2004)

⇒ continuation around saddle-node

- for Poincaré map by Misra *et al* 2008

Basic idea

Feedback control

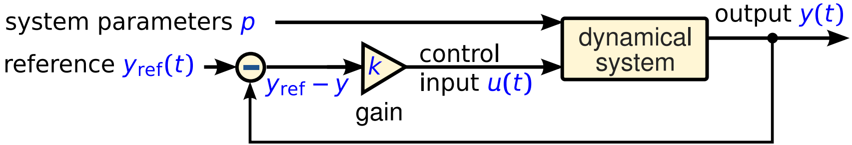


Find periodic orbits

- Time-delayed feedback (Pyragas): $y_{\mathrm{ref}} = y(t-T)$
- Extended (Gauthier): $y_{\mathrm{ref}} = (1-\varepsilon)y_{\mathrm{ref}}(t-T) + \varepsilon y(t-T)$
- T period, $0 < \varepsilon \leq 1$ ⇒ system with delay
- periodic orbits of period T with control = periodic orbits of period T without control
- stabilizing gains $k(t)$ exist for single input

Basic idea

Feedback control



General approach

- k stabilizing $\Rightarrow$ $y(t) \to y_\infty(t;\ y_{\text{ref}}, p)$ for $t \to \infty$
- **Input-Output Map**: $(y_{\text{ref}}, p) \mapsto y_\infty$
- **Solve** (e.g. Newton iteration)
  $$y_{\text{ref}} = y_\infty(y_{\text{ref}}, p)$$
  for y_{ref} and p in $\mathbb{R}^n$ or function space
  (few Fourier modes for y_{ref})
- $u \to 0$ for $t \to \infty$ in solution
- branch points equal branch points of uncontrolled system

Controlled experiment = fixed point map

- assume experiment with controllable equilibrium y_* (location unknown), and stabilising feedback

$$u(t) = k^T[y_{\mathrm{ref}} - y(t)]$$

⇒this defines **nonlinear input output map**

- $y_\infty : (y_{\mathrm{ref}}, p) \mapsto \lim_{t\to\infty} y(t)$
- One evaluation of y_∞:
  1. set system parameters to p,
  2. set input to feedback law to $u(t) = k^T[y_{\mathrm{ref}} - y(t)]$ (y is state/output)
  3. Wait until transients have settled: $y_\infty(y_{\mathrm{ref}}, p) := \lim_{t\to\infty} y(t)$
- y_{ref} is equilibrium of uncontrolled experiment if and only if

$$y_\infty(y_{\mathrm{ref}}, p) = y_{\mathrm{ref}} \quad \text{(which implies} \quad \lim_{t\to\infty} u(t) = 0\text{)}$$

Controlled experiment = fixed point map

- For **periodic orbit** $y_*(t)$ and stabilising feedback

$$u(t) = k^T[y_{\mathrm{ref}}(Tt) - y(t)]$$

⇒**nonlinear input output map** with $[0,1]$-periodic y_{ref}

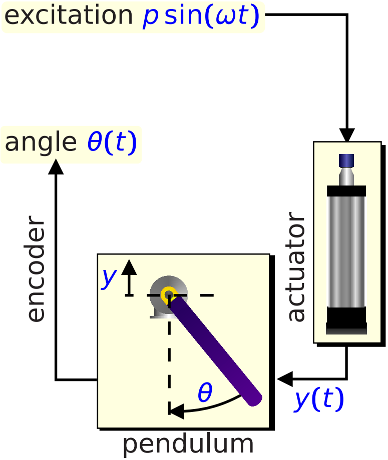
- $y_\infty : (y_{\mathrm{ref}}(T\cdot), T, p) \mapsto \lim_{n\to\infty} y(nT + T\cdot)$
- One evaluation of y_∞:
  1. set system parameters to p,
  2. set input to feedback law to $u(t) = k^T[y_{\mathrm{ref}}(t) - y(t)]$
  3. Wait until transients have settled:
     $y_\infty(y_{\mathrm{ref}}, T, p)(s) := \lim_{n\to\infty} y(nT + Ts)$
- y_{ref} is periodic orbit of uncontrolled experiment if and only if

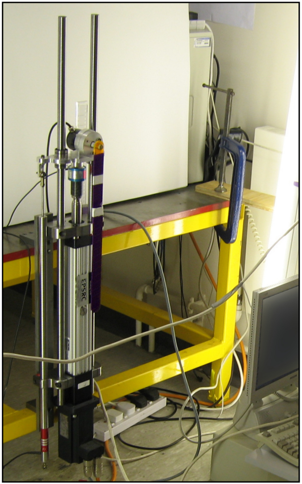
$$y_\infty(y_{\mathrm{ref}}, T, p) = y_{\mathrm{ref}} \quad \text{(which implies} \quad \lim_{t\to\infty} u(t) = 0)$$

- impose phase condition, pseudoarclength condition on y_{ref}, T, p
- (e.g.) represent y_{ref} by its first Fourier modes

Periodically forced example: rotation in pendulum

Schema





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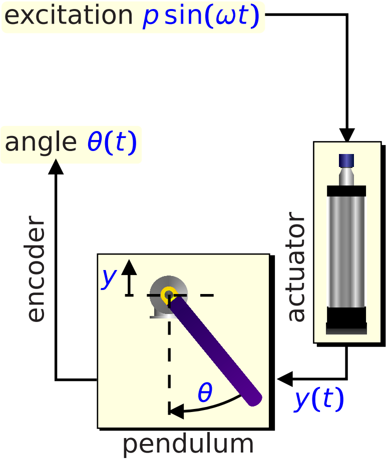
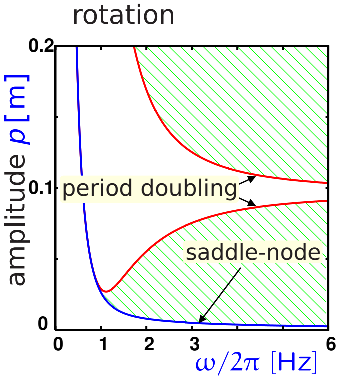


Numerics



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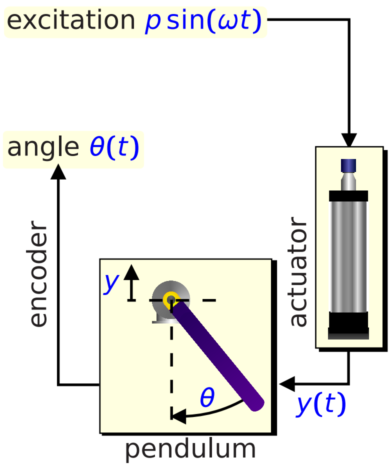

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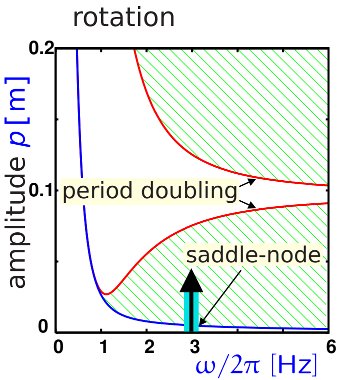

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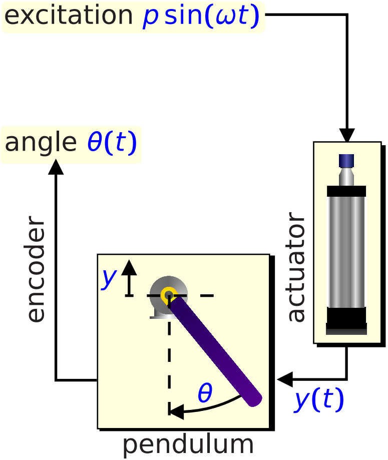
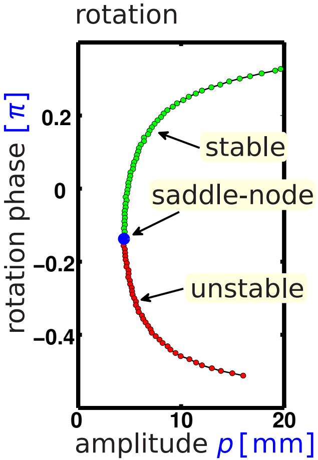


Experiment



Periodically forced example: rotation in pendulum

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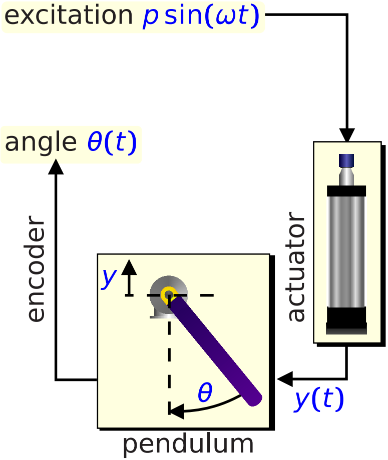

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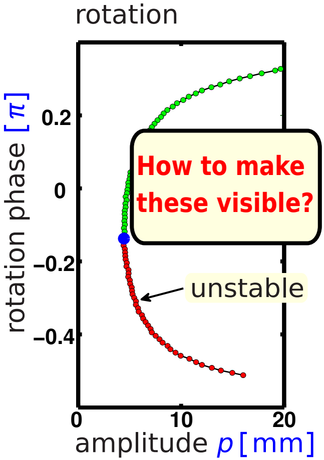

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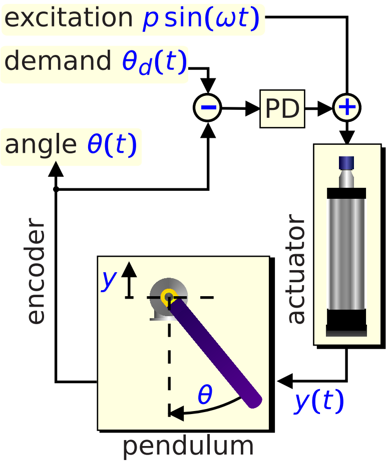


Control

- feed back $\theta - \theta_d$

⇒ always stable

⇒ output not natural

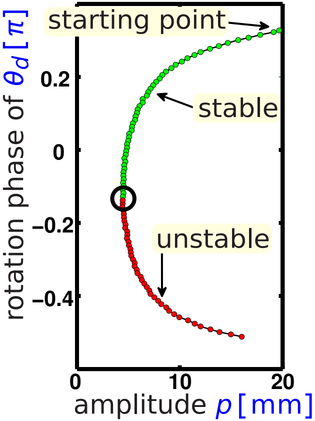
- How to choose θ_d?

⇒ find θ_d
such that $\theta_d - \theta = 0$

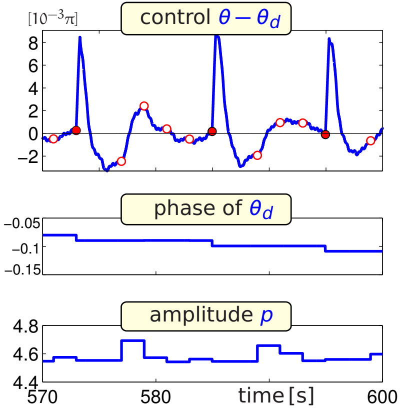
Experimental results

Experiment

bifurcation diagram



time profile of continuation



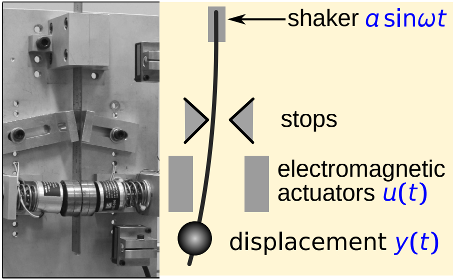
Experimental results

video

Mechanical experiments (periodically forced)

Backlash/impact oscillator
(Results from Bureau *et al*, 2014)

- CONTINEX (COCO) numerical continuation toolbox by F Schilder
- discretization of y_{ref}: 5 Fourier modes



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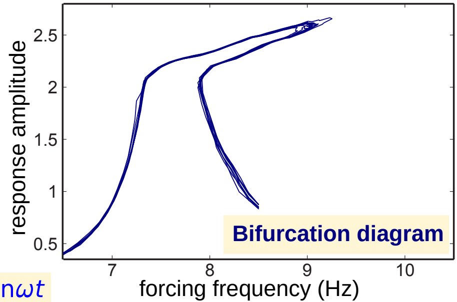

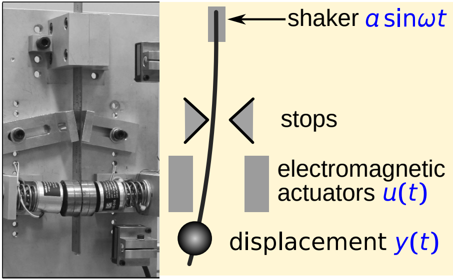

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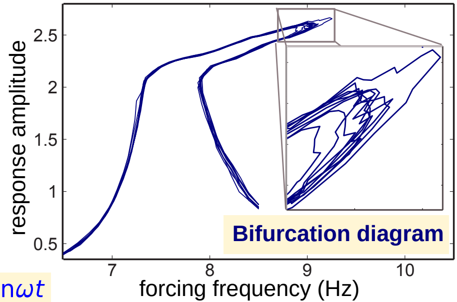

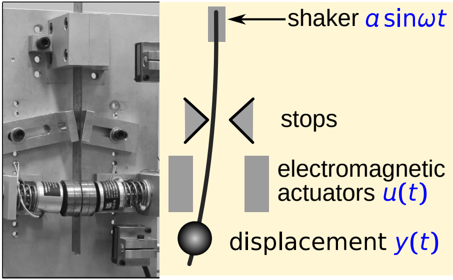

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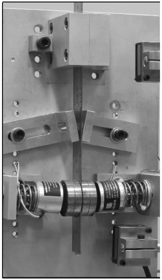

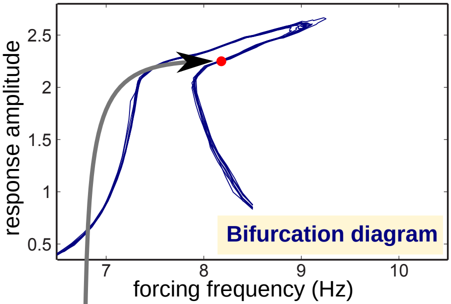

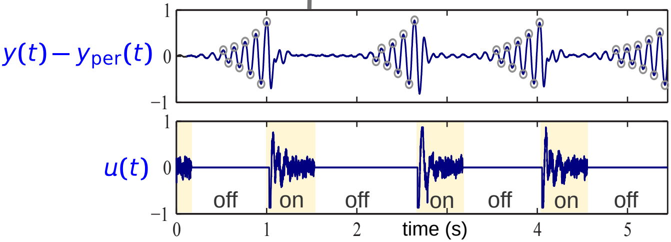

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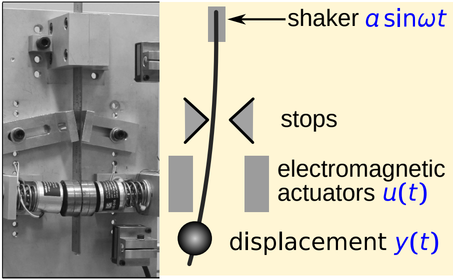

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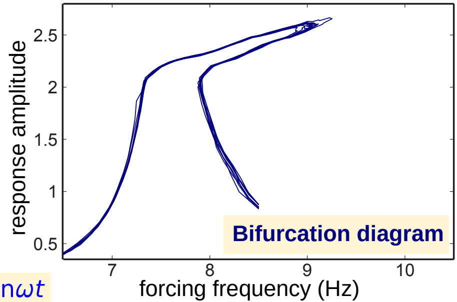

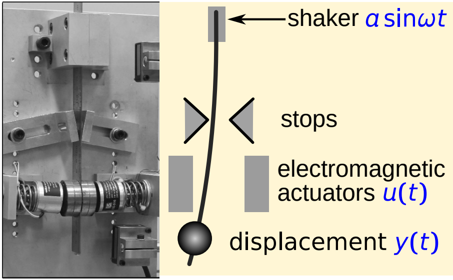

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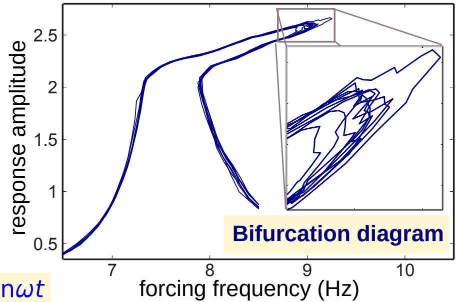

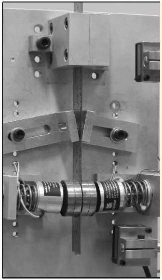

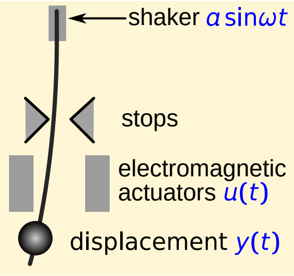

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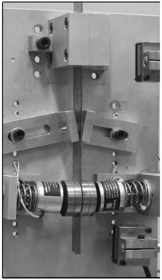

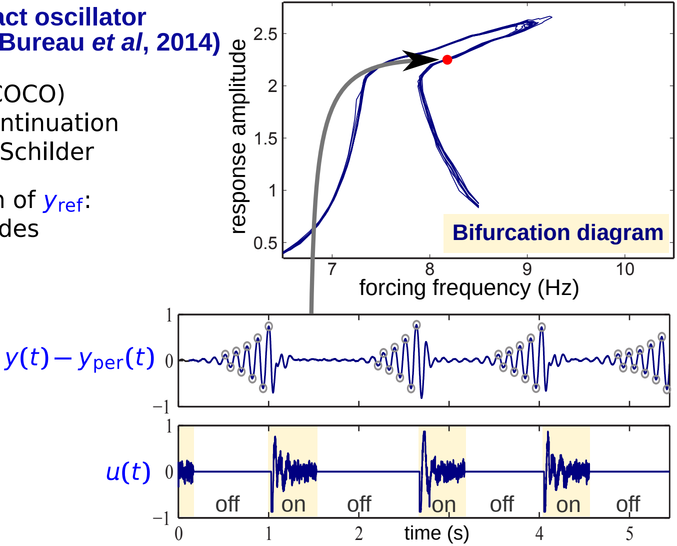

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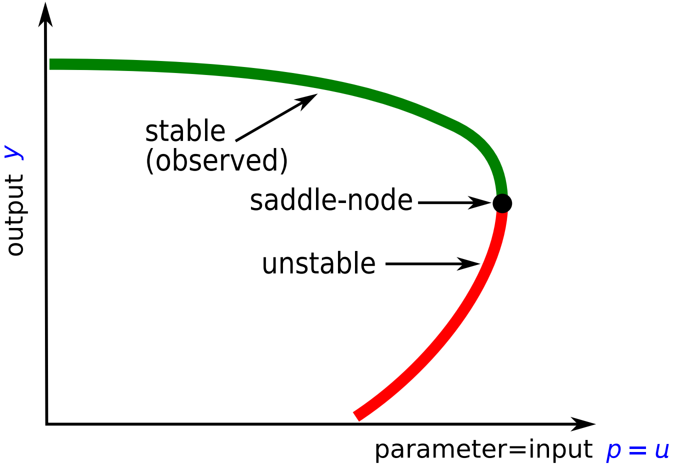
Simplification of *Siettos et al* (2004)

input $u(t)$ is bifurcation parameter p
output $y(t)$



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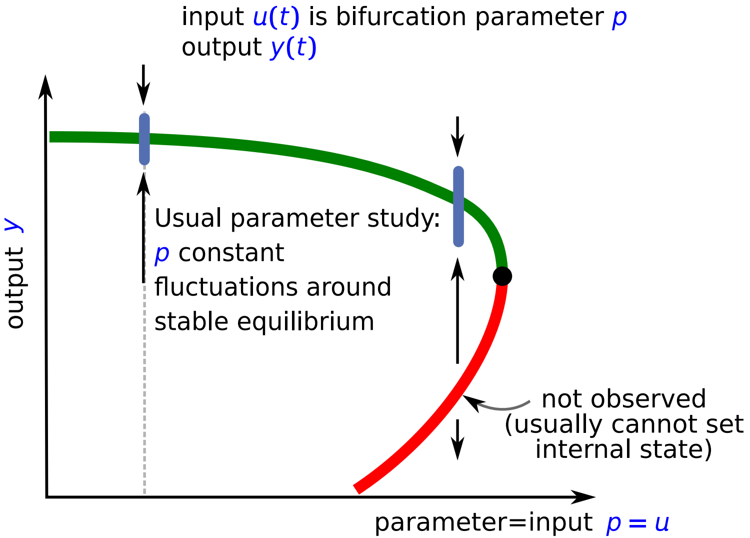

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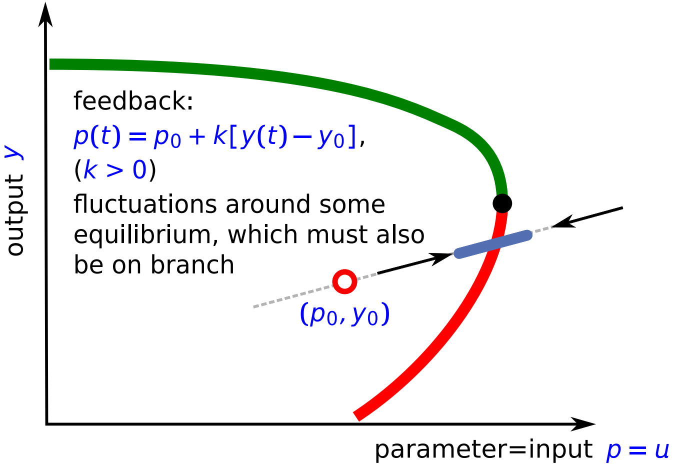

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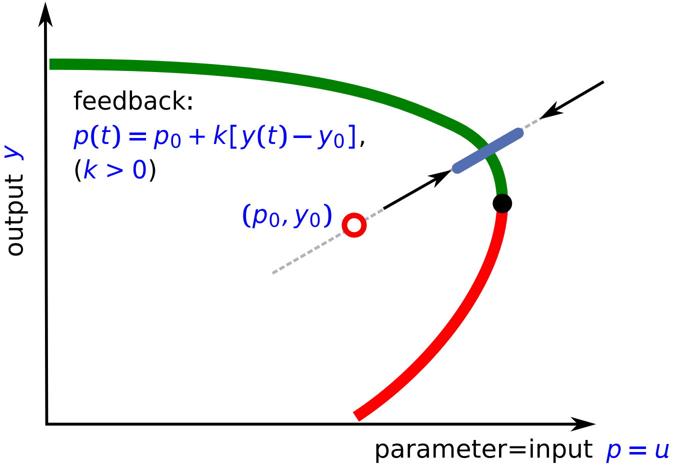

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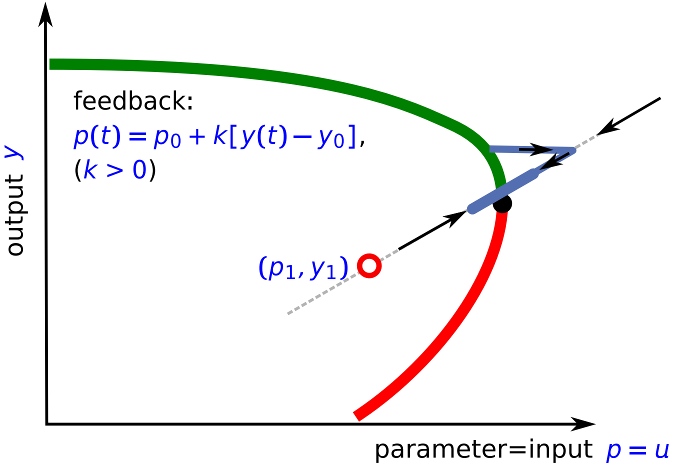

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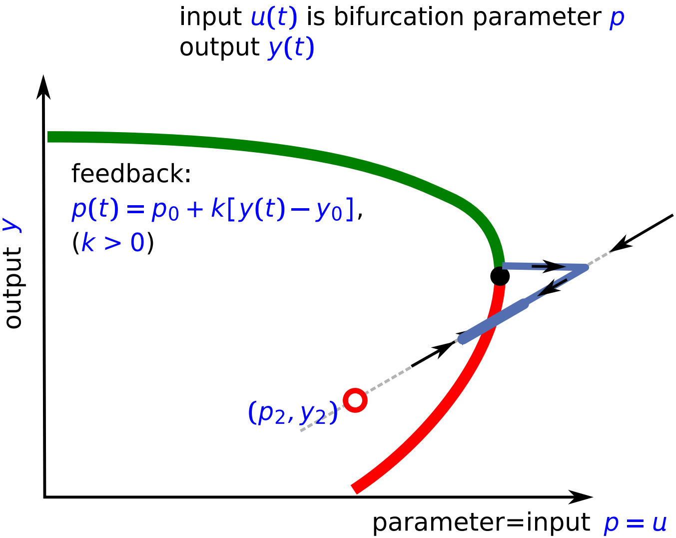

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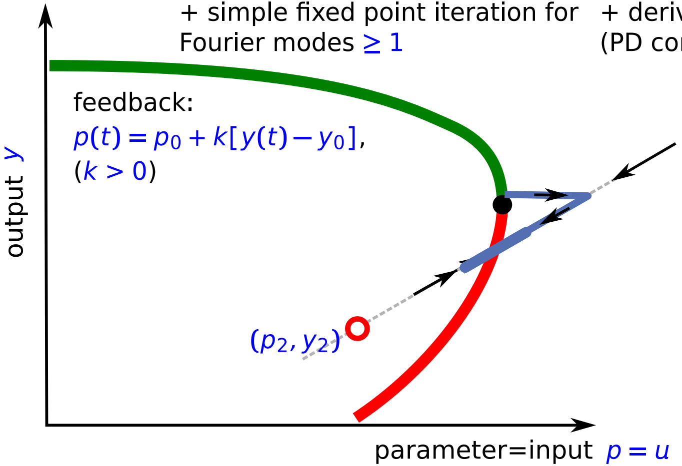


Motivating example IV

Check for saddle-node normal form:

$$\dot{y} = -p - y^2$$

$\Rightarrow$Equilibrium $y_s = \sqrt{-p}$ stable, $y_u = -\sqrt{-p}$ unstable.

$$p(t) = p_0 + k[y(t) - y_0]$$

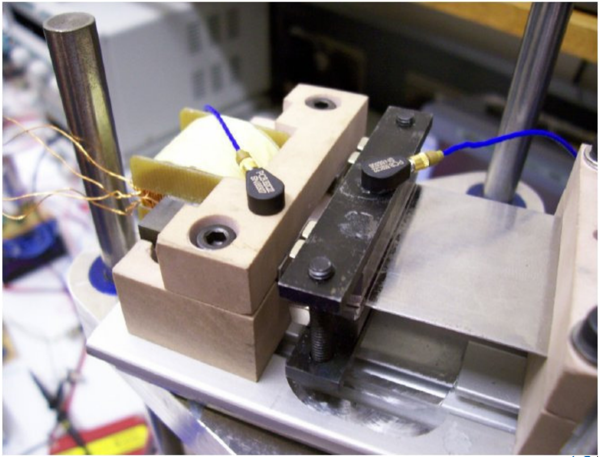
Equilibria satisfy

$$\begin{aligned} 0 &= -p_{\text{eq}} - y_{\text{eq}}^2 \\ &= -\{p_0 + k[y_{\text{eq}} - y_0]\} - y_{\text{eq}}^2 \end{aligned}$$

Stability: $-k - 2y_{\text{eq}}$ $\quad\Rightarrow\quad$ stable for $y_{\text{eq}} \geq -k/2$.

Energy harvester — *Barton & S*, 2013

nonlinear oscillator (damping difficult to model)



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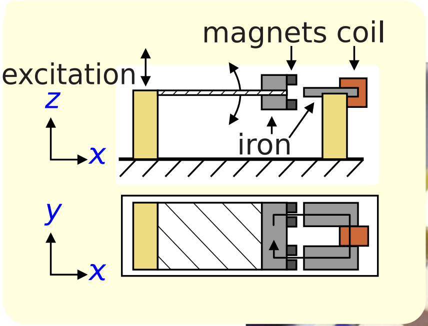
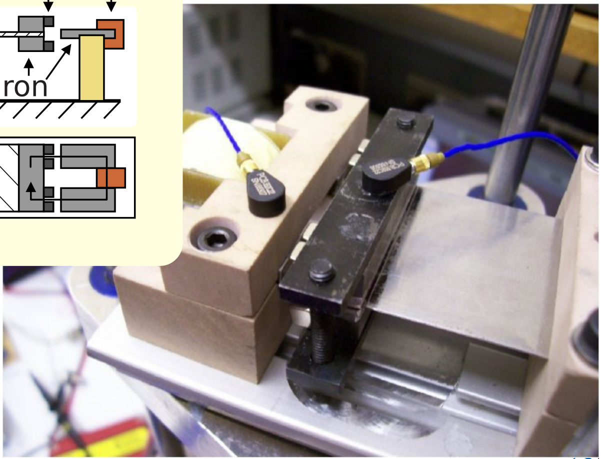
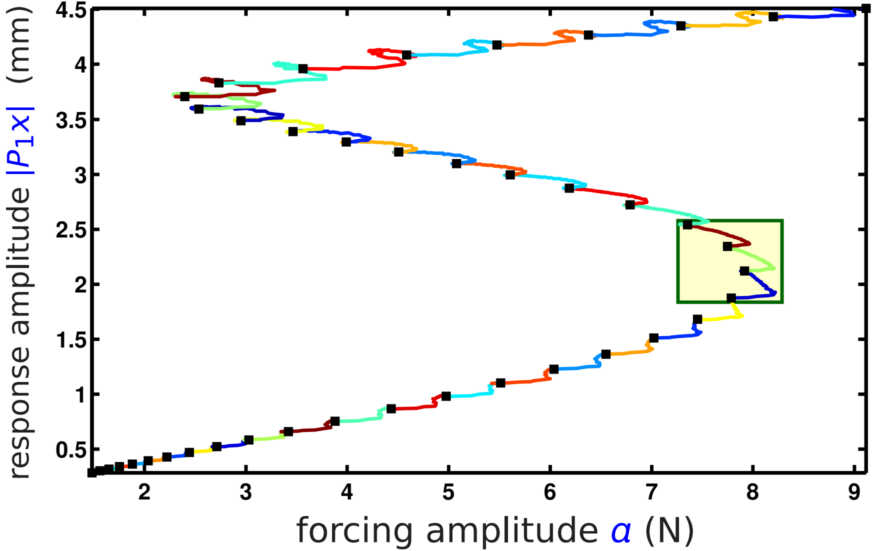




Evolution of control through bifurcation diagram



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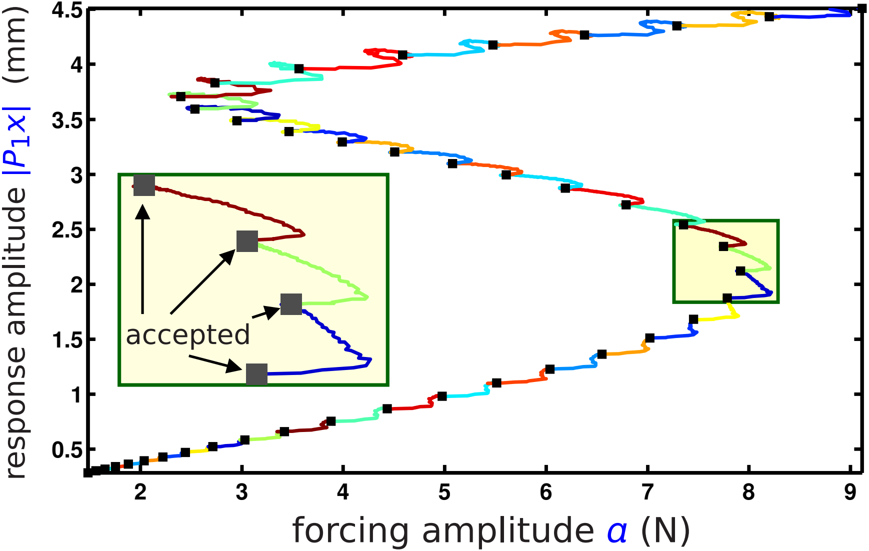

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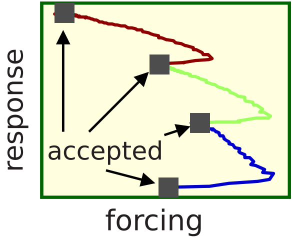
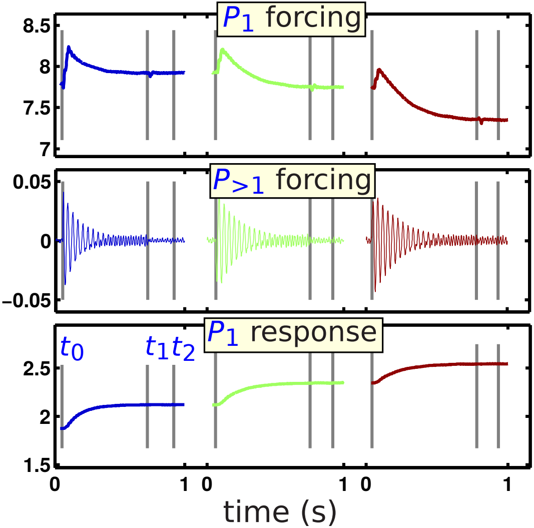
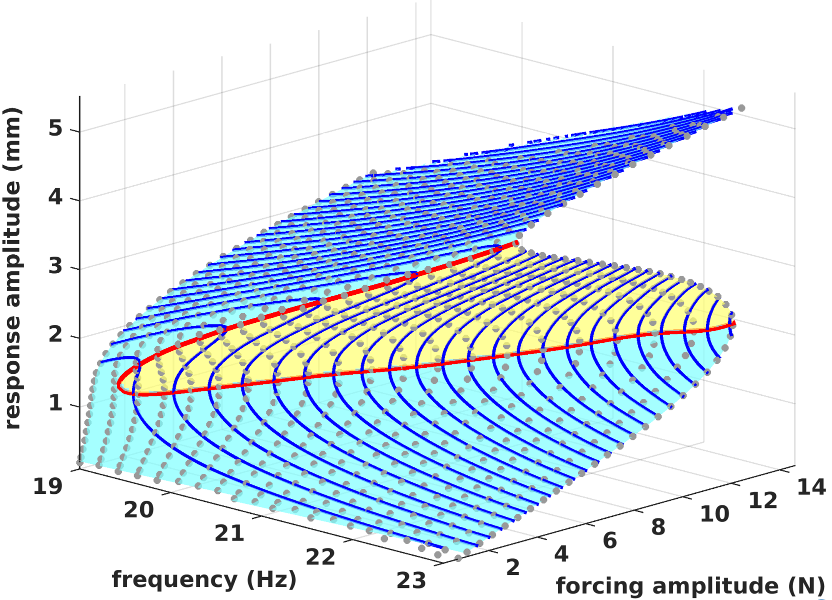
Time profiles





t_0	set new reference
t_1	control has settled
$\Rightarrow$	iterate reference
t_2	control has settled

Solution surface and errors



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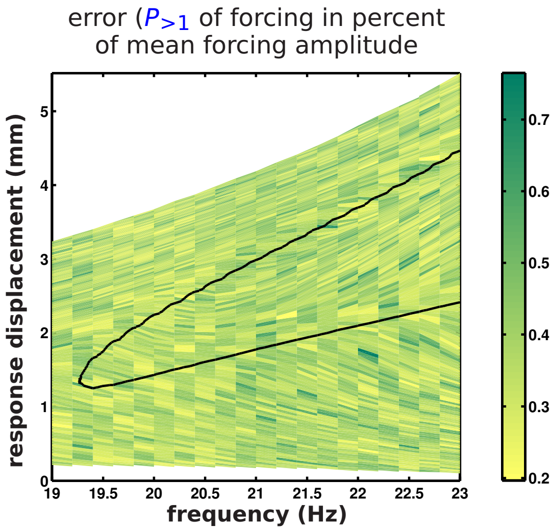
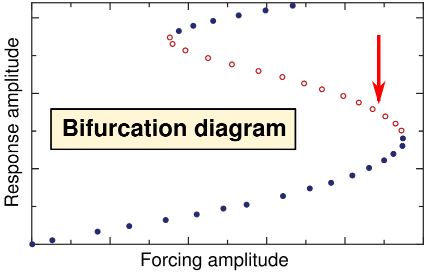


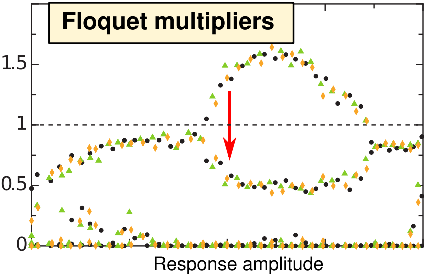
main source of error derivative component of PD control (greatly reduced with filtering)

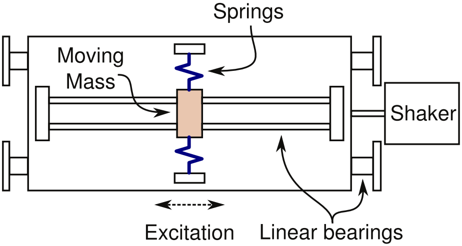
Stable and unstable subspaces of periodic orbits

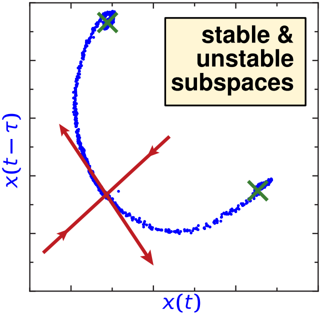
DAW Barton'17

arxiv:1506.04052









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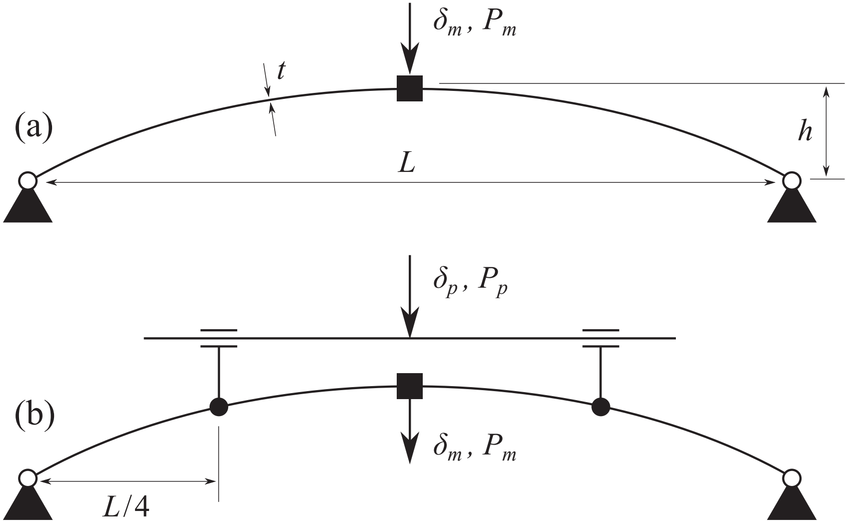
Recent experiments on buckling beams

Neville, Groh, Pirrera, Schenk, PRL'18



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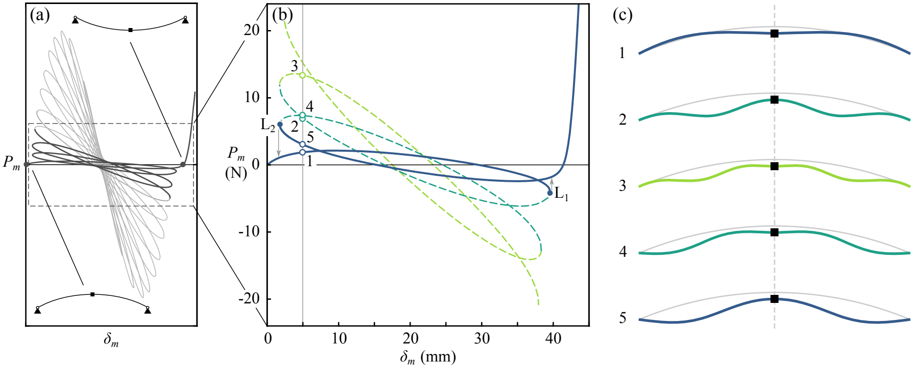
Numerical results for model



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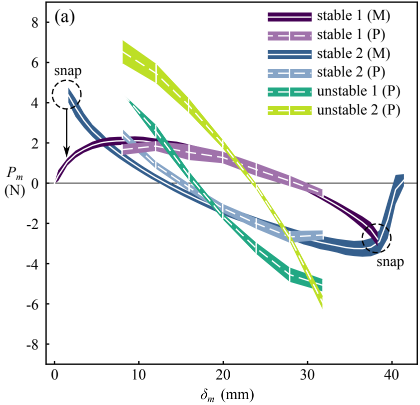
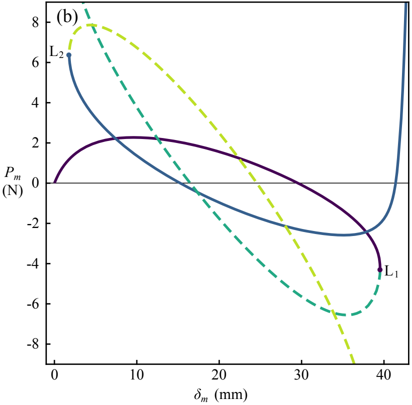
Experiment



Model



Using control-based continuation in simulations

Disease spreading on network

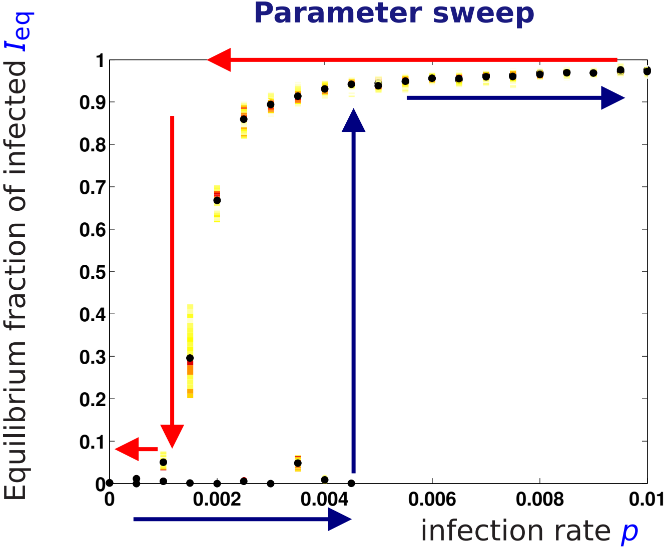
[Gross *et al*, PRL 2006]

- network with N nodes (individuals) with state either S (susceptible) or I (infected)
- kN links (initially random, $k \sim 10$)
- at every step:
  - I individual recovers with probability r
  - infection travels along SI link (infects S node) with probability p
  - SI link is rewired (keep S node, replace I node by random other S node) with probability w
- system has parameter range where disease-free and endemic equilibrium coexist

Disease spreading on network

[Gross *et al*, PRL 2006]

Parameter sweep



Disease spreading on network

[Gross *et al*, PRL 2006]

Mean field model

- I number of infected,
- L_{ab} number of links between a and b
- p infection probability, r recovery probability, w rewiring probability

$$\dot{I} = pL_{SI} - rI$$
$$\dot{L}_{II} = pL_{ISI} + pL_{SI} - 2rL_{II}$$
$$\dot{L}_{SS} = (r+w)L_{SI} - 2pL_{SSI}$$

$$N = I + S$$
$$kN = L_{II} + L_{SS} + L_{SI}$$

$L_{abc} \approx L_{ab}L_{bc}/b$ ⇐closure assumption

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Rigorous definition of equilibrium?

- almost surely disease goes extinct eventually
- avoid this effect by re-introducing infected nodes
- for p in bistable regime stochastic system has stationary density with (in projection) 2 well-separated local maxima **stable equilibria**:
  - disease close to extinct
  - large number of infected
- Consider macroscopic output, e.g., I, conditional balance in the long run

$$I_{\mathrm{eq}} = \lim_{t\to\infty} \operatorname{mean}\left\{I(s) : s \in [t, t+T], I(s) \in [I_{\mathrm{low}}, I_{\mathrm{up}}]\right\}$$

(expectation of conditional stationary density)

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Proposed definition of (unstable) equilibrium

Include feedback loop:

- Choose reference fraction of infected I_{ref}
- at every step:

  if $I < I_{\text{ref}}$,   infect $I_{\text{ref}} - I$ individuals
  if $I > I_{\text{ref}}$,   "cure" $I - I_{\text{ref}}$ individuals

- I_{ref} is equilibrium value if, after transients have settled

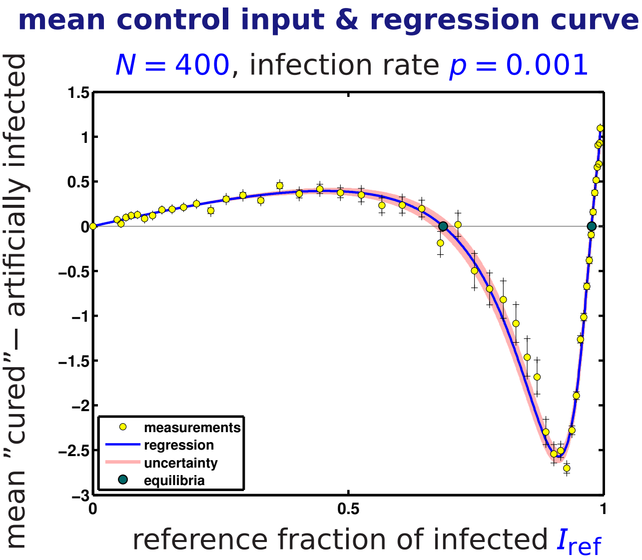
  $$\text{mean artificially cured} = \text{mean artificially infected}$$

- Or: long-time mean of feedback loop input is zero.

  $$\lim_{t\to\infty} E[I(t) - I_{\text{ref}}] = 0$$

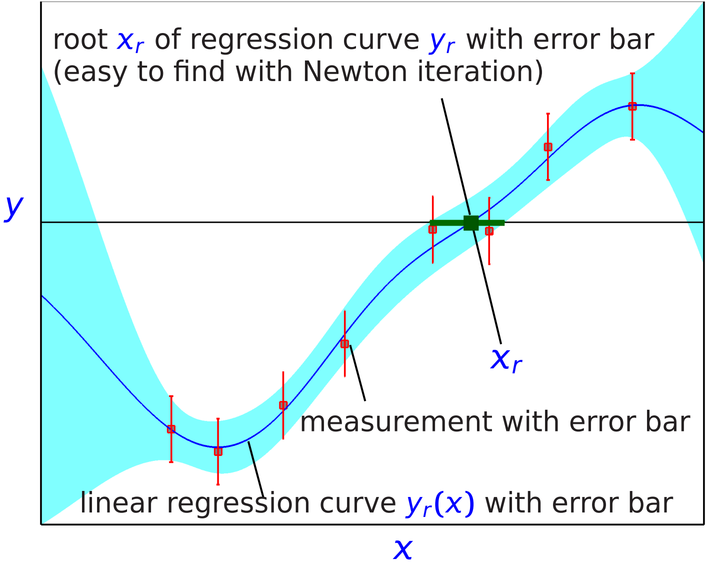
- For large N: $\lim_{t\to\infty} E[I(t) - I_{\text{ref}}]^2 \sim N^{-1}$.
- Resulting equilibrium I_{ref} independent of choice of feedback loop for $N \to \infty$.

Proposed definition of (unstable) equilibrium



Newton iteration & continuation with uncertainty

Linear regression with Gaussian process



Procedure for continuation with uncertainty

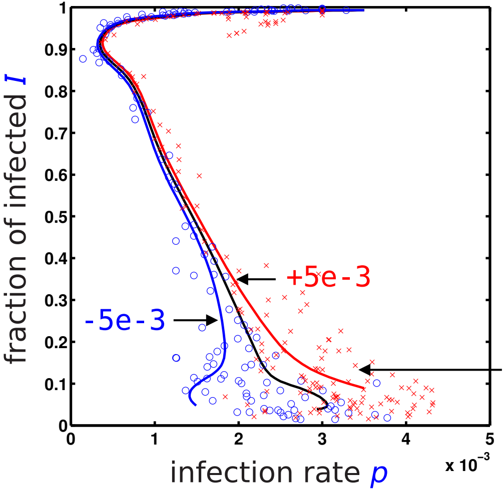
Inner loop (replacing Newton iteration):

1. find root x_0 of regression curve $y_r(x)$ with Newton it.
2. determine where to measure next:
   - x for which measurement $y_r(x) \pm \sigma_r(x)$ would minimize error bar of root x_0 for updated y_r, or
   - x where measurement $y_r(x) \pm \sigma_r(x)$ changes root x_0 the most

   Both are nonlinear optimization problems on current regression curve y_r (cheap in principle).

   optimal new x not necessary, only sensible x
3. stop if expected effect on root x_0 is not worth additional measurement.
4. measure $(\text{mean}\, y, \text{var}\, y)$ at new x, update (y_r, σ_r) with new value
5. back to 1.

Disease on network equilibrium continuation



$N = 400$

× mean $u > 0$

○ mean $u < 0$

close to tanscritical:
long transients

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Disease on network equilibrium continuation

Animation of fold continuation

Contination in experiments

- evaluation of residual $F : (y_{\text{ref}}, p) \to y_\infty$ is slow

⇒ restriced to low dimension of control inputs and small number of (eg) Fourier modes

- low accuracy of F (relative error $\approx 10^{-2}$ in clean experiments)

⇒ restricted to well-conditioned problems

⇒ F. Schilder's `coco` toolbox `continex`

- limiting factor: ability to provide stabilizing real-time feedback

⇒ hardest part is problem specific

⇒ new algorithms needed

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